

# ON GALILEAN CONNECTIONS AND THE FIRST JET BUNDLE

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ABSTRACT. We see how the first jet bundle of curves into affine space can be realized as a homogeneous space of the Galilean group. Cartan connections with this model are precisely the geometric structure of second-order ordinary differential equations under time-preserving transformations – sometimes called KCC-theory. With certain regularity conditions, we show that any such Cartan connection induces “laboratory” coordinate systems, and the geodesic equations in this coordinates form a system of second-order ordinary differential equations. We then show the converse – the “fundamental theorem” – that given such a coordinate system, and a system of second order ordinary differential equations, there exists regular Cartan connections yielding these, and such connections are completely determined by their torsion.

## 1. INTRODUCTION

The geometry of a system of ordinary differential equations has had a distinguished history, dating back even to Lie [10]. Historically, there have been three main branches of this theory, depending on the class of allowable transformations considered. The most studied has been differential equations under contact transformation; see §7.1 of Doubrov, Komrakov, and Morimoto [5], for the construction of Cartan connections under this class of transformation.

Another classical study has been differential equations under point-transformations. (See, for instance, Tresse [13]). By Bäcklund’s Theorem, this is novel only for a single second-order differential equation in one independent variable. The construction of the Cartan connection for this form of geometry was due to Cartan himself [2]. See, for instance, §2 of Kamran, Lamb and Shadwick [6], for a modern “equivalence method” treatment of this case.

The third case of transformations is the most applicable to classical mechanics; one considers point-transformation preserving the direction and flow of time. The invariants of a system of second-differential equations in this case were first analyzed by Kosambi [7, 8], with comment by Cartan [3]. Later, Chern applied the equivalence method to rigorously complete the classification of invariants [4]. Due to the contribution of these three authors, this form of geometry is sometimes called *KCC-theory*.

Here, we examine the Cartan connection for the third branch of this theory. Our technique is quite close to Cartan’s in [3], or for a similar but modern treatment see §8.3 of Sharpe [12]. The crux being that, with a slight change in notation, the structure equation of KCC-theory given by Chern in [4] become precisely the curvature equation for a Cartan connection modeled on the Galilean group.

Here, we pose the foundations of the global geometry of systems of second order differential equations in the language of Cartan connections. Formally, the flat version of this geometry is phase space (technically the first jet bundle of curves into affine space). We then “curve” this by introducing a space with Cartan connection modelled on the Galilean group, hereafter termed a “Galilean connection”. Unlike in the case of contact or point-transformations (which essentially have projective character), we are generally unable to produce a “good” coordinate system in which to work. The main result of our paper is that the imposition of an integrability condition allows us to prove the existence of laboratory coordinates (Theorem 3.3) adapted to our Galilean geometry. From this result, it follows that the geodesics of the system are solutions to a system of second order ordinary differential equations (Proposition 3.6).

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We then extend the “fundamental theorem” as given by Chern in [4] to include torsion. Specifically, we show that if given some coordinate system and a system of second order ordinary differential equations in these coordinates, then given any choice of tensors drawn from the appropriate space, there exists a unique Galilean connection with these coordinates as its laboratory frame, the system of differential equations as its geodesics, and the given tensors as its torsion.

The language of jet spaces we use sparingly; see for instance Saunders [11] for a full treatment of jet bundles. On the other hand we will use Cartan connections throughout this work. Sharpe [12] provides a good modern introduction to this theory. For a prerequisite, we indicate his §5.3 for Cartan connections, and Theorem 1.7.1 for the notion of the development of a curve. If the model space has the notion of a straight line, then it is natural to define a geodesic of a Cartan connection to be a curve whose development is a straight line; this is Sharpe’s Definition 6.2.6 for Riemannian geometry and Definition 8.3.4 for projective geometry. We will use this idea to define a geodesic of a Galilean geometry below.

Finally, we would like to thank the referees for making several points that greatly improve the presentation of this paper.

## 2. THE FIRST JET SPACE

Our goal is to realize the first jet space  $J^1(\mathbb{R}, \mathbb{R}^n)$  as a homogeneous space associated to a natural automorphism group. As a manifold, we have  $J^1(\mathbb{R}, \mathbb{R}^n) = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ ; if  $(t, x, v)$  is a (global) coordinate system adapted to this product structure, then  $J^1(\mathbb{R}, \mathbb{R}^n)$  comes equipped with a preferred family of “straight” lines:  $\{(t, vt, v)\}_{t \in \mathbb{R}}$  for any fixed  $v \in \mathbb{R}^n$ . This is the geometric content of Newton’s first law. For our purposes, we need the following description for a straight line.

**Lemma 2.1.** *Let  $\sigma: \mathbb{R} \rightarrow J^1(\mathbb{R}, \mathbb{R}^n)$ . Then  $\sigma$  is a straight line if and only if  $\sigma^*(dt) \neq 0$ , and  $\sigma^*(dx - v dt) = \sigma^*(dv) = 0$ .*

Any natural automorphism group of  $J^1(\mathbb{R}, \mathbb{R}^n)$  will be required to carry straight lines to straight lines (these are called contact transformations). In other words, we consider local diffeomorphisms preserving the exterior ideal  $\langle dx - v dt, dv \rangle$ . There are a number of possible automorphism groups, depending on how much of the product structure is preserved:

- Contact:  $(\bar{t}, \bar{x}, \bar{v}) = (\bar{t}(t, x, v), \bar{x}(t, x, v), \bar{v}(t, x, v))$ ,
- Point:  $(\bar{t}, \bar{x}, \bar{v}) = (\bar{t}(t, x), \bar{x}(t, x), \bar{v}(t, x, v))$ ,
- Fibre-preserving:  $(\bar{t}, \bar{x}, \bar{v}) = (\bar{t}(t), \bar{x}(t, x), \bar{v}(t, x, v))$ , and
- Galilean:  $(\bar{t}, \bar{x}, \bar{v}) = (t + t_0, \bar{x}(t, x), \bar{v}(t, x, v))$ .

As noted in the preface, we are interested in the last of these groups of transformations, which in our opinion is the most applicable to classical mechanics. It is fair to call these “Galilean” transformations because of the following result.

**Proposition 2.2.** *Let  $\Phi(t, x, v)$  be a Galilean transformation of  $J^1(\mathbb{R}, \mathbb{R}^n)$ . Then there exists constant  $t_0 \in \mathbb{R}$ ,  $x_0, v_0 \in \mathbb{R}^n$ , and  $A \in Gl(n)$  such that*

$$\Phi(t, x, v) = (t + t_0, Ax + v_0 t + x_0, Av + v_0).$$

*Proof.* Let us write  $\Phi(t, x, v) = (\bar{t}, \bar{x}, \bar{v})$  for simplicity. We have  $\bar{t} = t + t_0$  for some constant  $t_0 \in \mathbb{R}$  by assumption. Now,

$$\begin{aligned} d\bar{x}^i - \bar{v}^i d\bar{t} &= \frac{\partial \bar{x}^i}{\partial x^j} dx^j + \frac{\partial \bar{x}^i}{\partial t} dt - \bar{v}^i dt \\ &= \frac{\partial \bar{x}^i}{\partial x^j} (dx^j - v^j dt) + \left( \frac{\partial \bar{x}^i}{\partial x^j} v^j - \bar{v}^i + \frac{\partial \bar{x}^i}{\partial t} \right) dt. \end{aligned}$$

Since the ideal  $\langle dx - v dt, dv \rangle$  is preserved, we must have

$$\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} v^j + \frac{\partial \bar{x}^i}{\partial t}. \quad (1)$$

By the same reasoning,

$$\begin{aligned} d\bar{v}^i &= \frac{\partial \bar{v}^i}{\partial v^j} dv^j + \frac{\partial \bar{v}^i}{\partial x^j} dx^j + \frac{\partial \bar{v}^i}{\partial t} dt \\ &= \frac{\partial \bar{v}^i}{\partial v^j} dv^j + \frac{\partial \bar{v}^i}{\partial x^j} (dx^j - v^j dt) + \left( \frac{\partial \bar{v}^i}{\partial x^j} v^j + \frac{\partial \bar{v}^i}{\partial t} \right) dt, \end{aligned}$$

gives

$$\frac{\partial \bar{v}^i}{\partial x^j} v^j + \frac{\partial \bar{v}^i}{\partial t} = 0.$$

Substituting from (1) above,

$$\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} v^j v^k + 2 \frac{\partial^2 \bar{x}^i}{\partial x^j \partial t} v^j + \frac{\partial^2 \bar{x}^i}{\partial t \partial t} = 0.$$

This must hold for all  $v$ -and the coefficients are independent of  $v$ -so:

$$\begin{aligned} \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} = 0 & \quad \text{implies} \quad \bar{x} = A(t)x + b(t); \\ \frac{\partial^2 \bar{x}^i}{\partial x^j \partial t} = 0 & \quad \text{implies in above} \quad A = \text{constant}; \text{ and} \\ \frac{\partial^2 \bar{x}^i}{\partial t \partial t} = 0 & \quad \text{implies in above} \quad b(t) = v_0 t + x_0. \end{aligned}$$

Therefore,  $\bar{x} = Ax + v_0 t + x_0$  and by (1),  $\bar{v} = Av + v_0$ . □

**Corollary 2.3.** *The Galilean transformations act transitively on the first jet space.*

### 3. GALILEAN GEOMETRY

Strictly speaking, we are not considering the usual Galilean group of classical mechanics as we allow for any linear map—rather than just rotations—in addition to the boosts and translations in time and space.<sup>1</sup> There is a convenient representation of the Galilean group as matrices

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ x & v & A \end{pmatrix} : t \in \mathbb{R}, x, v \in \mathbb{R}^n, A \in Gl(n) \right\}.$$

The isotropy group of the origin in the first jet bundle is

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} : A \in Gl(n) \right\},$$

giving us the first jet space as a homogeneous space,  $J^1(\mathbb{R}, \mathbb{R}^n) \cong G/H$ .

The canonical left-invariant Maurer–Cartan form for this group is easily computed,

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -A^{-1}(x - vt) & -A^{-1}v & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ dt & 0 & 0 \\ dx & dv & dA \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ dt & 0 & 0 \\ A^{-1}(dx - v dt) & A^{-1}dv & A^{-1}dA \end{pmatrix}. \end{aligned}$$

Note how the ideal  $\langle dx - v dt, dv \rangle$  appears canonically as part of the geometry.

**Definition 3.1.** A *Galilean manifold* is a smooth manifold  $X$ , equipped with an principal  $H$ -bundle,  $\pi: P \rightarrow X$ , and a Cartan connection on  $P$  with values in  $\mathfrak{g}$ .

<sup>1</sup>In what follows, one can use the traditional Galilean group, but the resulting geometry is slightly different. It is, in fact, related to that proposed in [9].

As indicated above, we will call the Cartan connection of a Galilean manifold a *Galilean connection*. To fix our notation, we will express our Galilean connection as  $\begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ \omega & \phi & \Pi \end{pmatrix}$ . The curvature of our connection then has structure equations

$$\begin{pmatrix} 0 & 0 & 0 \\ T & 0 & 0 \\ \Omega & \Phi & R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ d\tau & 0 & 0 \\ d\omega + \Pi \wedge \omega + \phi \wedge \tau & d\phi + \Pi \wedge \phi & d\Pi + \Pi \wedge \Pi \end{pmatrix}.$$

In the first jet space  $J^1(\mathbb{R}, \mathbb{R}^n)$ , we had global coordinate systems of the form  $(t, x, v)$  in which we described our straight lines. But, in the most general Galilean manifolds, such coordinates need not exist even locally. Curvature measures the failure of the geometric structure to be representable in coordinate systems. Therefore, we are really interested in Galilean connections for which some components of its curvature vanish. The second condition in the definition below is essentially the same notion as a regular Cartan connection in parabolic geometry, [1]). The first however is an integrability condition that, to the best of our knowledge, is new to this framework.

**Definition 3.2.** A *regular* Galilean manifold is a Galilean manifold whose Cartan connection satisfies  $T = 0$  and  $\Omega \equiv 0 \pmod{\omega}$ .

The curvature conditions of a regular Galilean manifold are precisely those required to yield the desired coordinates. The following theorem, which is the main result of this paper, is a precise statement of this fact. Intuitively, we have demanded certain curvatures vanish in order to find local coordinates which mimic the adapted coordinates of the first jet bundle.

**Theorem 3.3.** *Let  $X$  be a Galilean manifold, with Cartan connection as above. Then  $X$  is regular if and only if about any point of  $X$ , there exists a coordinate system,  $(t, x, v)$ , and a local section  $s$  of  $P$ , such that  $s^*\tau = dt$ ,  $s^*\omega = dx - v dt$ , and  $s^*\phi \equiv dv \pmod{dt, dx}$ .*

**Definition 3.4.** Coordinate systems  $(t, x, v)$  as in Theorem 3.3 are called *laboratory coordinates* of a regular Galilean manifold.

*Proof of Theorem 3.3.* Suppose  $X$  be a regular, and let  $\tilde{s}$  be any local section of  $P$  near our given point. Immediately we have  $d\tilde{s}^*\tau = 0$ , and hence restricting our domain if necessary  $\tilde{s}^*\tau = dt$ . As  $dt \neq 0$ , the function  $t$  serves as a coordinate function.

Define  $\mathcal{I}$  to be the exterior ideal on  $X$  generated by the one-forms  $dt$  and  $\tilde{s}^*\omega$ . The curvature conditions  $T = 0$  and  $\Omega \equiv 0 \pmod{\omega}$  show that  $\mathcal{I}$  is integrable. Since  $dt \in \mathcal{I}$  each integral manifold lies in a  $t = \text{constant}$  hypersurface. Again restricting our domain if necessary, we may choose independent functions  $x^j$  so that the integral manifolds of  $\mathcal{I}$  are uniquely described by  $t = \text{constant}$  and  $x = \text{constant}$ .

Note that the integral manifolds of  $\mathcal{I}$  are also the integral manifolds of  $\langle dt, dx^j \rangle$ . Therefore, at each point of our domain we may express  $\tilde{s}^*\omega = A(dx - v dt)$  for some invertible matrix  $A$  and vector  $v$ . We claim the functions  $\{v^j\}$  complement  $(t, x)$  to form a local coordinate system. To show this, it suffices to show that  $\{dt, dx, dv\}$  forms a local coframe near our given point. To this end, we compute:

$$d\tilde{s}^*\omega = dA \cdot A^{-1} \wedge \tilde{s}^*\omega - A dv \wedge dt.$$

However, as  $\Omega \equiv 0 \pmod{\omega}$ , we must have  $\Omega = \Phi \wedge \omega$  for some matrix-valued one-form  $\Phi$ . We then have the structure equation  $d\omega = (\Phi - \Pi) \wedge \omega - \phi \wedge \tau$ . Therefore,

$$(dA \cdot A^{-1} + \tilde{s}^*\Pi - \tilde{s}^*\Phi) \wedge \tilde{s}^*\omega = (A dv - \tilde{s}^*\phi) \wedge dt.$$

Every two-form in the left side of this equation contains a  $\tilde{s}^*\omega$ . Thus,  $A dv - \tilde{s}^*\phi = N\tilde{s}^*\omega + \lambda dt$  for some matrix  $N$  and vector  $\lambda$  (this is essentially Cartan's lemma). However,  $\tau, \omega, \phi$  are linearly independent on  $P$ . Therefore, the  $dv^j$  are linearly independent, and also independent from  $\tilde{s}^*\omega, \tilde{s}^*\tau$ , and thus independent from  $dt, dx$  as well.

Finally, define a new section by  $s = R_A \tilde{s}$ , where here  $A$  abbreviates  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix} \in H$ . Then we have  $s^* \tau = \tilde{s}^* \tau$ ,  $s^* \omega = A^{-1} \tilde{s}^* \omega$ , and  $s^* \phi = A^{-1} \tilde{s}^* \phi$ . In particular,  $s^* \tau = dt$ ,  $s^* \omega = dx - v dt$ , and  $s^* \phi \equiv dv \pmod{s^* \tau, s^* \omega}$ , as desired.

For the converse, we compute

$$s^* T = ds^* \tau = d(dt) = 0,$$

and writing  $s^* \phi = dv + N dx + \lambda dt$ , we have

$$\begin{aligned} s^*(\Omega - \Pi \wedge \omega) &= ds^* \omega + s^* \phi \wedge s^* \tau \\ &= N dx \wedge dt = N s^* \omega \wedge s^* \tau. \end{aligned}$$

Hence  $s^* \Omega \equiv 0 \pmod{s^* \omega}$ , and since  $\Omega$  is horizontal, we have  $\Omega \equiv 0 \pmod{\omega}$ .  $\square$

**Definition 3.5.** A *geodesic* in a Galilean manifold is a curve<sup>2</sup>  $\sigma: (-\epsilon, \epsilon) \rightarrow X$  whose development  $\tilde{\sigma}: (-\epsilon, \epsilon) \rightarrow J^1(\mathbb{R}, \mathbb{R}^n)$  is contained in a straight line.

That is,  $\sigma: (-\epsilon, \epsilon) \rightarrow X$  is a geodesic precisely if one (and hence every) lift  $\hat{\sigma}: (-\epsilon, \epsilon) \rightarrow P$  satisfies  $\hat{\sigma}^* \omega = \hat{\sigma}^* \phi = 0$ .

**Proposition 3.6.** Let  $X$  be a regular Galilean manifold,  $(t, x, v)$  be laboratory coordinates in  $X$ , and write  $s^* \phi = dv + \Gamma(t, x, v) dt + N(t, x, v)(dx - v dt)$ . Then the geodesics of  $X$  are the solutions to the system of second order differential equations

$$\frac{d^2 x^j}{dt^2} + \Gamma^j \left( t, x, \frac{dx}{dt} \right) = 0. \quad (2)$$

*Proof.* We have already seen that the straight lines in  $J^1(\mathbb{R}, \mathbb{R}^n)$  are characterized by the equations  $\tilde{\sigma}^*(dt) \neq 0$ ,  $\tilde{\sigma}^*(dx - v dt) = 0$ , and  $\tilde{\sigma}^*(dv) = 0$ . However,  $dt$ ,  $dx - v dt$ , and  $dv$  are the  $\tau$ ,  $\omega$ , and  $\phi$  terms, respectively, in the Maurer–Cartan forms of the Galilean group. Thus by the definition of development, a curve  $\sigma$  is a geodesic if and only if it is non-degenerate and  $\sigma^*(s^* \omega) = \sigma^*(s^* \phi) = 0$ . However, in laboratory coordinates  $s^* \omega = dx - v dt$  and  $s^* \phi = dv + \Gamma dt + N(dx - v dt)$ ; the integral curves to these forms are precisely the solutions to (2).  $\square$

We saw above that regular Galilean connections are precisely those with laboratory coordinates, and the geodesics of such connections in these coordinates are systems of second order differential equations. The converse of this is the foundational result of Chern, which we recall here using our notation and terminology.

**Theorem 3.7** (Chern, [4]). *Given a system of second order ordinary differential equations, (2), there exists a unique regular Galilean connection satisfying  $\Omega = 0$  and  $\Phi \equiv 0 \pmod{\omega}$  whose geodesics are given by these differential equations in laboratory coordinates.*

Chern proved this by a direct application of the method of equivalence, and since all torsions could be normalized to zero, this led to a particularly simple connection. With the machinery developed above, it is simple for us to classify all regular Galilean connections, as presented in the result below.

**Proposition 3.8.** *Any regular Galilean connection has*

$$\begin{aligned} T &= 0 \\ \Omega^i &= C^i_j \tau \wedge \omega^j + \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k + S^i_{jk} \omega^j \wedge \phi^k \\ \Phi^i &= (D^i_j + C^i_j) \tau \wedge \phi^j + Q^i_{jk} \omega^j \wedge \phi^k \\ &\quad + P^i_j \tau \wedge \omega^j + \frac{1}{2} R^i_{jk} \omega^j \wedge \omega^k - \frac{1}{2} (S^i_{jk} - S^i_{kj}) \phi^j \wedge \phi^k. \end{aligned}$$

<sup>2</sup>More accurately it is a germ of a curve as the  $\epsilon$  in the domain has not been specified.

Together with the geodesic equations, (2), the tensors  $D$ ,  $C$ ,  $S$ , and  $Q$  parameterize in a one-to-one way regular Galilean connections, in that, given an arbitrary smooth selection of these four tensors there exists a unique regular Galilean connection realizing them in its curvature.

*Proof.* Working in a laboratory coordinate system  $(t, x, v)$  and suppressing the section  $s$ , we have the components of a regular Galilean connection satisfy

$$\begin{aligned}\tau &= dt \\ \omega^i &= dx^i - v^i dt \\ \phi^i &= dv^i + \Gamma^i dt + N^i_j(dx^j - v^j dt) \\ \Pi^i_j &= \Lambda^i_j dt + \Gamma^i_{jk} \omega^k + \Delta^i_{jk} \phi^k.\end{aligned}$$

We have seen that  $\Gamma^i$  define the geodesic equations of our connection; the functions  $N^i_j$ ,  $\Lambda^i_j$ ,  $\Gamma^i_{jk}$ , and  $\Delta^i_{jk}$  can be arbitrary. From the structure equation  $\Omega = d\omega + \Pi \wedge \omega + \phi \wedge \tau$  we find

$$\begin{aligned}C^i_j &= \Lambda^i_j - N^i_j \\ T^i_{jk} &= \Gamma^i_{kj} - \Gamma^i_{jk} \\ S^i_{jk} &= -\Delta^i_{jk}.\end{aligned}$$

First, we compute

$$\Phi^i = d\phi^i + \Pi^i_j \wedge \phi^j \equiv \Delta^i_{jk} \phi^j \wedge \phi^k \pmod{\tau, \omega}.$$

Thus, the  $\phi \wedge \phi$  component of  $\Phi$  is as claimed. Computing the remaining terms in  $\Phi^i$  is routine, and so we merely state the relevant terms:

$$\begin{aligned}D^i_j + C^i_j &= \Lambda^i_j + N^i_j - \frac{\partial \Gamma^i}{\partial v^j} \\ Q^i_{jk} &= \Gamma^i_{kj} - \frac{\partial N^i_j}{\partial v^k}.\end{aligned}$$

From these, and the relations above, it is easy to solve for  $N^i_j$ ,  $\Lambda^i_j$ ,  $\Gamma^i_{jk}$ , and  $\Delta^i_{jk}$  in terms of  $D$ ,  $C$ ,  $S$ , and  $Q$ .  $\square$

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